【10920 趙啟超教授離散數學 / 第 15 堂版書】

EECS 2060 Discrete Mathematics
Spring 2021

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Homework Assignment No. 3
Due 10:10am, April 28, 2021

Reading: Grimaldi: Sections 10.1 The First-Order Linear Recurrence Relation, 10.2 The Second-Order Linear Homogeneous Recurrence Relation with Constant Coefficients, 10.3 The Nonhomogeneous Recurrence Relation, 9.1 Introductory Examples, 9.2 Definition and Examples: Calculational Techniques, 10.4 The Method of Generating Functions.

Problems for Solution:

1. Solve the recurrence relation

$$a_{n+2} + 4a_{n+1} + 8a_n = 0, \qquad n \geq 0$$

with initial conditions $a_0 = 0$ and $a_1 = 2$.

2. Solve the recurrence relation

$$a_{n+2} - 6a_{n+1} + 9a_n = 3 \cdot 2^n + 7 \cdot 3^n, \qquad n \geq 0$$

with initial conditions $a_0 = 1$ and $a_1 = 4$.

3. In this problem the recurrence relation will be used to find a formula for $a_n =$ the sum of the first n cubes. That is, $a_1 = 1^3$, $a_2 = 1^3 + 2^3$, $a_3 = 1^3 + 2^3 + 3^3, \ldots$. Find the recurrence relation that a_n satisfies (with appropriate initial condition) and then solve for it.

4. Suppose your parents would like to get a mortgage (loan) of C dollars from the bank to buy a new house, at an *annual* interest rate r for a period of N years. The usual practice is to repay the mortgage in equal *monthly* installments of D dollars each. You, as a student of EECS 2060, should be able to compute the value of D, which is a function of C, r, and N, for your parents. Please find the value of D. (*Hint:* An annual interest rate r is equivalent to a monthly interest rate $r/12$, and currently r is around 1.3% to 1.8% for mortgage in Taiwan. There will be a total of $12N$ monthly installments for a period of N years, and N is typically 20 or 30 now in Taiwan. Let a_n represent the *unpaid balance* after n monthly payments have been made. Then just before the $(n+1)$th payment, the new balance will be $(1 + r/12) \cdot a_n$, and just after the $(n+1)$th payment the unpaid balance will be $(1+r/12)a_n - D$. Thus the sequence $\{a_n\}$ satisfies the recurrence relation: $a_{n+1} = (1 + r/12)a_n - D$.)

5. Use the generating function method to solve the recurrence relation

$$a_n - a_{n-1} - 2a_{n-2} = 2^n, \quad n \geq 2$$

with initial conditions $a_0 = 4$ and $a_1 = 12$.

6. Let F_n, $n \geq 0$, be the Fibonacci numbers. The *Lucas numbers* L_n can be defined by

$$L_n = F_{n+1} + F_{n-1}, \quad \text{for } n \geq 1$$

with $L_0 = 2$. Find the generating function for L_n.

7. Consider the following system of recurrence relations:

$$\begin{aligned} a_n &= -2a_{n-1} - 4b_{n-1} \\ b_n &= 4a_{n-1} + 6b_{n-1} \end{aligned}$$

for $n \geq 1$, with initial conditions $a_0 = 1$ and $b_0 = 0$.

   (a) Find the generating function for a_n and then solve for a_n.

   (b) Do the same for b_n.

8. Consider the system of recurrence relations in Problem 7.

   (a) Find the recurrence relation that a_n satisfies (with appropriate initial conditions).

   (b) Do the same for b_n.

Homework Collaboration Policy: I allow and encourage discussion or collaboration on the homework. However, you are expected to write up your own solution and understand what you turn in. Late homework is subject to a penalty of 5% to 40% of your total points.

Generating Functions for Solving Recurrence Relations

Example Fibonacci numbers

$$F_n = F_{n-1} + F_{n-2},\quad n \geq 2 \quad \text{with } F_0 = 0,\ F_1 = 1$$

Let the generating function for F_n be $F(x)$

$$\sum_{n\geq 2} F_n x^n - \sum_{n\geq 2} F_{n-1} x^n - \sum_{n\geq 2} F_{n-2} x^n = 0$$

Fibonacci numbers

$$F_n = F_{n-1} + F_{n-2},\quad n \geq 2 \quad \text{with } F_0 = 0,\ F_1 = 1$$

Let the generating function for F_n be $F(x)$

$$\sum_{n\geq 2} F_n x^n - \sum_{n\geq 2} F_{n-1} x^n - \sum_{n\geq 2} F_{n-2} x^n = 0$$

$$\sum_{n\geq 2} F_n x^n = F_2 x^2 + F_3 x^3 + \cdots = F(x) - F_0 - F_1 x$$

$$\sum_{n\geq 2} F_{n-1} x^n = F_1 x^2 + F_2 x^3 + \cdots$$

$$= x\,(F_1 x + F_2 x^2 + \cdots) = x\,(F(x) - F_0)$$

$$\sum_{n\geq 2} F_{n-2} x^n = F_0 x^2 + F_1 x^3 + \cdots$$

$$= x^2 (F_0 + F_1 x + \cdots) = x^2 F(x)$$

Generating Functions for Solving Recurrence Relations

Example Fibonacci numbers

$$F_n = F_{n-1} + F_{n-2},\ n \geq 2 \quad \text{with } F_0 = 0,\ F_1 = 1$$

Let the generating function for F_n be $F(x)$

$$\sum_{n\geq 2} F_n x^n - \sum_{n\geq 2} F_{n-1} x^n - \sum_{n\geq 2} F_{n-2} x^n = 0$$

$$\sum_{n\geq 2} F_n x^n = F_2 x^2 + F_3 x^3 + \cdots = F(x) - F_0 - F_1 x$$

$$\begin{aligned}\sum_{n\geq 2} F_{n-1} x^n &= F_1 x^2 + F_2 x^3 + \cdots \\ &= x\left(F_1 x + F_2 x^2 + \cdots\right) = x\left(F(x) - F_0\right)\end{aligned}$$

$$\begin{aligned}\sum_{n\geq 2} F_{n-2} x^n &= F_0 x^2 + F_1 x^3 + \cdots \\ &= x^2\left(F_0 + F_1 x + \cdots\right) = x^2 F(x)\end{aligned}$$

$$\Rightarrow \left(F(x) - F_0 - F_1 x\right) - x\left(F(x) - F_0\right) - x^2 F(x) = 0$$

$$\Rightarrow F(x)\left(1 - x - x^2\right) = F_0 + (F_1 - F_0)x = x$$

$$\begin{aligned}\Rightarrow F(x) &= \frac{x}{1 - x - x^2} = \frac{x}{(1-\lambda_1 x)(1-\lambda_2 x)} \\ &= \frac{\alpha_1}{1-\lambda_1 x} + \frac{\alpha_2}{1-\lambda_2 x} \quad \text{(partial fraction expansion)}\end{aligned}$$

$$\boxed{\lambda^n \rightleftharpoons \frac{1}{1-\lambda x}}$$

where $\lambda_1 = \frac{1+\sqrt{5}}{2}$ and $\lambda_2 = \frac{1-\sqrt{5}}{2}$

$$\Rightarrow \quad x = \alpha_1 (1-\lambda_2 x) + \alpha_2 (1-\lambda_1 x)$$

Let $x = \lambda_1^{-1}$. $\quad \lambda_1^{-1} = \alpha_1 (1-\lambda_2 \lambda_1^{-1})$

$$\Rightarrow \quad \alpha_1 = \frac{1}{\lambda_1 - \lambda_2} = \frac{1}{\sqrt{5}}.$$

Let $x = \lambda_2^{-1}$. $\quad \lambda_2^{-1} = \alpha_2 (1-\lambda_1 \lambda_2^{-1})$

$$\Rightarrow \quad \alpha_2 = \frac{1}{\lambda_2 - \lambda_1} = -\frac{1}{\sqrt{5}}.$$

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$$\Rightarrow \alpha_2 = \frac{1}{\lambda_2-\lambda_1} = -\frac{1}{\sqrt{5}}.$$

$$\Rightarrow (F(x)-F_0-F_1x) - x(F(x)-F_0) - x^2F(x) = 0$$

$$\Rightarrow F(x)(1-x-x^2) = F_0 + (F_1-F_0)x = x$$

$$\Rightarrow F(x) = \frac{x}{1-x-x^2} = \frac{x}{(1-\lambda_1 x)(1-\lambda_2 x)}$$

$$= \frac{\alpha_1}{1-\lambda_1 x} + \frac{\alpha_2}{1-\lambda_2 x} \quad \text{(partial fraction expansion)}$$

$$\boxed{\lambda^n \rightleftharpoons \frac{1}{1-\lambda x}}$$

where $\lambda_1 = \frac{1+\sqrt{5}}{2}$ and $\lambda_2 = \frac{1-\sqrt{5}}{2}$

$$F(x) = \frac{\frac{1}{\sqrt{5}}}{1-\lambda_1 x} - \frac{\frac{1}{\sqrt{5}}}{1-\lambda_2 x}.$$

Therefore, $\quad F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right], \quad n \geq 0.$

Example $\quad a_n - 3a_{n-1} = n, \; n \geq 1 \;$ with $a_0 = 1$.

Let the generating function for a_n be $A(x)$.

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$$\Rightarrow (A(x) - a_0) - 3xA(x) = \sum_{n\geq 1} n x^n = \frac{x}{(1-x)^2}$$

$$\boxed{n\lambda^n \rightleftharpoons \frac{\lambda x}{(1-\lambda x)^2}}$$

$$\Rightarrow (1-3x)A(x) = a_0 + \frac{x}{(1-x)^2}$$

$$\Rightarrow A(x) = \frac{1}{1-3x} + \frac{x}{(1-x)^2(1-3x)}$$

Let $\frac{x}{(1-x)^2(1-3x)} = \frac{\alpha_1}{1-x} + \frac{\alpha_2}{(1-x)^2} + \frac{\alpha_3}{1-3x}$

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$$F(x) = \frac{\frac{1}{\sqrt{5}}}{1-\lambda_1 x} - \frac{\frac{1}{\sqrt{5}}}{1-\lambda_2 x}.$$

Therefore, $F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$, $n \geq 0$.

Example $a_n - 3a_{n-1} = n$, $n \geq 1$ with $a_0 = 1$.

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Let $\frac{x}{(1-x)^2(1-3x)} = \frac{\alpha_1}{1-x} + \frac{\alpha_2}{(1-x)^2} + \frac{\alpha_3}{1-3x}$

$$\Rightarrow x = \alpha_1(1-x)(1-3x) + \alpha_2(1-3x) + \alpha_3(1-x)^2$$

Let $x = 1$. $1 = \alpha_2(1-3) \Rightarrow \alpha_2 = -\frac{1}{2}$.

Let $x = \frac{1}{3}$. $\frac{1}{3} = \alpha_3\left(1-\frac{1}{3}\right)^2 \Rightarrow \alpha_3 = \frac{3}{4}$.

Let $x = 0$. $0 = \alpha_1 + \alpha_2 + \alpha_3 \Rightarrow \alpha_1 = -\frac{1}{4}$.

$$\Rightarrow \quad x = \alpha_1 (1-x)(1-3x) + \alpha_2 (1-3x) + \alpha_3 (1-x)^2$$

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Let $x=0$. $\quad 0 = \alpha_1 + \alpha_2 + \alpha_3 \Rightarrow \alpha_1 = -\frac{1}{4}$.

Hence $A(x) = \frac{1}{1-3x} + \frac{-\frac{1}{4}}{1-x} + \frac{-\frac{1}{2}}{(1-x)^2} + \frac{\frac{3}{4}}{1-3x}$

$= \frac{\frac{7}{4}}{1-3x} + \frac{-\frac{1}{4}}{1-x} + \frac{-\frac{1}{2}}{(1-x)^2} \qquad \left((n+1)\lambda^n \rightleftharpoons \frac{1}{(1-\lambda x)^2} \right)$

Therefore, $a_n = \frac{7}{4} 3^n - \frac{1}{4} - \frac{1}{2}(n+1)$

$= \frac{7}{4} 3^n - \frac{1}{2} n - \frac{3}{4}, \quad n \geq 0.$

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Example $a_n - 2a_{n-1} - a_{n-2} + 2a_{n-3} = \begin{cases} 1, & \text{for } n=3 \\ 0, & \text{for } n \geq 4 \end{cases}$

with $a_0 = a_1 = 0$, $a_2 = 2$.

Let the generating function for a_n be $A(x)$.

$$(A(x) - a_0 - a_1x - a_2x^2) - 2x(A(x) - a_0 - a_1x) - x^2(A(x) - a_0) + 2x^3A(x) = 1\cdot x^3 + 0\cdot x^4 + 0\cdot x^5 + \dots = x^3$$

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$$\Rightarrow \quad A(x)(1 - 2x - x^2 + 2x^3) = 2x^2 + x^3$$

$$\Rightarrow \quad A(x) = \frac{2x^2 + x^3}{1 - 2x - x^2 + 2x^3} = x^2 \cdot \frac{2 + x}{1 - 2x - x^2 + 2x^3}$$

Let $\dfrac{2 + x}{1 - 2x - x^2 + 2x^3} = \dfrac{\alpha_1}{1 - 2x} + \dfrac{\alpha_2}{1 - x} + \dfrac{\alpha_3}{1 + x}$

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$$\left(A(x) - a_0 - a_1x - a_2x^2\right) - 2x\left(A(x) - a_0 - a_1x\right) - x^2\left(A(x) - a_0\right) + 2x^3A(x) = 1\cdot x^3 + 0\cdot x^4 + 0\cdot x^5 + \cdots = x^3$$

$$\Rightarrow \quad A(x)\left(1 - 2x - x^2 + 2x^3\right) = 2x^2 + x^3$$

$$\Rightarrow \quad A(x) = \frac{2x^2 + x^3}{1 - 2x - x^2 + 2x^3} = x^2 \cdot \frac{2+x}{1 - 2x - x^2 + 2x^3}$$

Let $\dfrac{2+x}{1 - 2x - x^2 + 2x^3} = \dfrac{\alpha_1}{1-2x} + \dfrac{\alpha_2}{1-x} + \dfrac{\alpha_3}{1+x}$

Then
$$\alpha_1 = \left.\frac{2+x}{(1-x)(1+x)}\right|_{x=\frac{1}{2}} = \frac{\frac{5}{2}}{\frac{1}{2}\cdot\frac{3}{2}} = \frac{10}{3}$$

$$\alpha_2 = \left.\frac{2+x}{(1-2x)(1+x)}\right|_{x=1} = \frac{3}{-1\cdot 2} = -\frac{3}{2}$$

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$$\Rightarrow \quad A(x) = x^2\left(\frac{\frac{10}{3}}{1-2x} - \frac{\frac{3}{2}}{1-x} + \frac{\frac{1}{6}}{1+x}\right)$$

$$\therefore \quad a_n = \frac{10}{3}\, 2^{n-2} - \frac{3}{2} + \frac{1}{6}(-1)^{n-2}, \quad \text{for } n \geq 2.$$

Then $\alpha_1 = \left.\frac{2+x}{(1-x)(1+x)}\right|_{x=\frac{1}{2}} = \frac{\frac{5}{2}}{\frac{1}{2}\cdot\frac{3}{2}} = \frac{10}{3}$

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$$\therefore \quad a_n = \frac{10}{3}\, 2^{n-2} - \frac{3}{2} + \frac{1}{6}(-1)^{n-2}, \quad \text{for } n \geq 2.$$

Generating Function for Enumeration

Let A and B be two sets of nonnegative integers. Suppose n is a nonnegative integer.

Question: How many solutions are there to the equation

$$a+b=n$$

Let A and B be two sets of nonnegative integers. Suppose n is a nonnegative integer.

Question: How many solutions are there to the equation

$$a+b=n$$

such that $a \in A$ and $b \in B$.

Example $A = \{0,1,2,3,4\}$

$B = \{2,3,4\}$

$2 = 0+2$

$3 = 0+3 = 1+2$

$4 = 0+4 = 1+3 = 2+2$

$5 = 1+4 = 2+3 = 3+2$

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Example

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n	0	1	2	3	4	5	$\cdots$
g_n	0	0	1	2	3	3	$\cdots$

Let this number be g_n and the generating function for g_n be

$$G(x) = g_0 + g_1 x + g_2 x^2 + \cdots$$

The generating function for A is

$$G_A(x) = \sum_{a \in A} x^a$$

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and the generating function for B is

$$G_B(x) = \sum_{b \in B} x^b$$

Consider $\left(\sum_{a \in A} x^a \right) \left(\sum_{b \in B} x^b \right)$

A typical term in this product is of the form

$$x^a \cdot x^b = x^{a+b}$$

x^5

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g_n	0	0	1	2	3	3	...

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A typical term in this product is of the form

$$x^a \cdot x^b = x^{a+b}$$

x^3

$\therefore$ The coefficient of x^n in $G_A(x)\,G_B(x)$ is the number of ways of choosing $a \in A$ and $b \in B$ such that $a+b=n$, i.e., g_n.

Therefore, $G_A(x)\,G_B(x) = G(x)$.

In general, let g_n denote the number of solutions (in nonnegative integers) to

$$a_1 + a_2 + \cdots + a_m = n$$

such that $a_i \in A_i$, $i = 1, 2, \ldots, m$.

Then $G(x) = G_{A_1}(x)\,G_{A_2}(x) \cdots G_{A_m}(x)$

where $G(x)$ is the generating function for g_n and $G_{A_i}(x)$ is the generating function for A_i, $i = 1, 2, \ldots, m$.

$$\Rightarrow \quad x = \alpha_1(1-\lambda_2 x) + \alpha_2(1-\lambda_1 x)$$

Let $x = \lambda_1^{-1}$. $\quad \lambda_1^{-1} = \alpha_1(1-\lambda_2\lambda_1^{-1})$

$$\Rightarrow \alpha_1 = \frac{1}{\lambda_1-\lambda_2} = \frac{1}{\sqrt{5}}.$$

Let $x = \lambda_2^{-1}$. $\quad \lambda_2^{-1} = \alpha_2(1-\lambda_1\lambda_2^{-1})$

$$\Rightarrow \alpha_2 = \frac{1}{\lambda_2-\lambda_1} = -\frac{1}{\sqrt{5}}.$$

: The coefficient of x^n in $G_A(x)\,G_B(x)$ is the number of ways of choosing $a \in A$ and $b \in B$ such that $a+b=n$, i.e., g_n.

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$B = \{2, 3, 4\}$

$2 = 0+2$

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Example (Cont.) $A = \{0, 1, 2, 3, 4\}$

$B = \{2, 3, 4\}$

$G_A(x) = 1 + x + x^2 + x^3 + x^4$

$G_B(x) = x^2 + x^3 + x^4$

$G(x) = G_A(x)\, G_B(x) = (1+x+x^2+x^3+x^4)(x^2+x^3+x^4)$

$= x^2 + 2x^3 + 3x^4 + 3x^5 + 3x^6 + 2x^7 + x^8$

Example Find the number of solutions in nonnegative integers to $z_1 + z_2 + \cdots + z_m = n$.

Let this number be g_n.

We have $A_1 = \{0, 1, 2, \ldots\} = A_2 = \cdots = A_m$.

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$$G(x) = \sum_{n \geq 0} g_n x^n = G_{A_1}(x)\, G_{A_2}(x) \cdots G_{A_m}(x)$$
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$$= (1-x)^{-m}$$
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$\therefore$ $g_n = \binom{m+n-1}{n}$, $n = 0, 1, 2, \cdots$

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with $4 \le z_2 \le 7$, $2 \le z_3 \le 6$, $z_4 \ge 13$.

Let $A_1=\{0,1,2,3,\dots\}$, $A_2=\{4,5,6,7\}$

$A_3=\{2,3,4,5,6\}$, $A_4=\{13,14,15,\dots\}$

Then

$$G_{A_1}(x) = 1+x+x^2+x^3+\dots = \frac{1}{1-x}.$$

$$G_{A_2}(x) = x^4+x^5+x^6+x^7.$$

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$2 = 0 + 2$

$3 = 0 + 3 = 1 + 2$

$4 = 0 + 4 = 1 + 3 = 2 + 2$

$5 = 1 + 4 = 2 + 3 = 3 + 2$

The generating function for g_n is

$$G(x) = G_{A_1}(x)\, G_{A_2}(x)\, G_{A_3}(x)\, G_{A_4}(x)$$

$$= \frac{x^{19}(1 + x + x^2 + x^3)(1 + x + x^2 + x^3 + x^4)}{(1-x)^2}$$